XVIII. *On some new Methods of investigating the Sums of several Classes of infinite Series. By* Charles Babbage, *Esq. A. M. F. R. S.*

Read April 1, 1819.

The processes which it is the object of this paper to explain, were discovered several years since; but certain difficulties connected with the subject, which I was at that time unable to explain, and which were equally inexplicable to several of my friends, to whom I had communicated these methods, induced me to defer publishing them, until I could offer some satisfactory solution.

These observations refer more particularly to the second method which I have detailed in this paper, and which may not inappropriately be called the *method of expanding horizontally and summing vertically*. Some traces of this method may, perhaps, be found in former writers, and particularly in a paper by Professor Vince, "On the Summation of Series," printed in the Philosophical Transactions for 1791; but there exists this peculiarity in that which I have employed, that after a certain number of the vertical columns are summed, all the remainder either vanish, or else have some common factor. This method, which I employed about the year 1812, gave the values of a variety of series whose sums had not hitherto been known, most of which were apparently correct, but some of the consequences which followed were evidently erroneous. About this time, Mr. Herschel, to whom I had communicated these anomalous results, by following a very different course, arrived at several general theorems, which, when applied to the series I had obtained, gave the same results. This coincidence at first increased my confidence in the values so discovered, and I continued to examine the reason why my own formulæ were in some cases defective. Mr. HERSCHEL's method was published in the Philosophical Transactions for 1814; and it was not until some time after that I perceived, that although the investigations were very different, the fundamental principle was the same in both methods. This induced me to attempt summing the same series by a direct process, and I succeeded in obtaining their sums by integration relative to finite differences, aided by certain peculiar artifices. The results obtained by this new plan, which is the first treated of in this paper, coincided with those already found, and seemed to confirm their truth, without in the least indicating the cause of the error: this cause however I now began to suspect, and, after some enquiry, I was at length able to detect. I have found that the *method of expanding horizontally and summing vertically*, will always lead to correct results, provided a certain series which I have pointed out, is finite. I have also shown how to express this series by a definite integral; and when this integral or this series has a finite value, the method may be depended on. In case this series or this definite integral is not finite, then the value of the series* multiplied by zero, must be added to

* The investigation of this series is generally a task of considerable difficulty. I have however given an example, wherein the correction thus found, added to the sum indicated by the method we are considering, gives the true value of the series, which in this case is one whose sum has been found by Euler.

the sum given by this method. In this latter case, however, the mode of summation which I have proposed, is not well adapted for giving the sums of series; its greatest advantage is felt when the integral or series alluded to is finite: but even in this case the criterion I have pointed out is not useless, for it serves to except certain particular values of the variables, which would give incorrect results. Without this criterion, or without something equivalent to it, I am inclined to think that the principle on which this method is founded, although it will probably in many cases give accurate results, will in others produce such as are not only numerically but symbolically untrue. It is worthy of remark, that the *method of expanding horizontally and summing vertically*, in many instances, gives precisely the same formulæ as the direct process of integration; yet that that method attaches limitations to them, which are necessary to their accuracy, but which are not indicated by the method last mentioned.

Before I proceed to explain these two processes, it will be convenient to prove that the values of all series of the forms

$$\underset{1}{A}x\frac{(\sin\theta)^m}{(\cos\theta)^n}+\underset{2}{A}x^2\frac{(\sin 2\theta)^m}{(\cos 2\theta)^n}+\underset{3}{A}x^3\frac{(\sin 3\theta)^m}{(\cos 3\theta)^n}+\&c.$$

$$\underset{1}{A}x\frac{(\cos\theta)^m}{(\sin\theta)^n}+\underset{2}{A}x^2\frac{(\cos 2\theta)^m}{(\sin 2\theta)^n}+\underset{3}{A}x^3\frac{(\cos 3\theta)^m}{(\sin 3\theta)^n}+\&c.$$

depend on series of the form

$$\underset{1}{A}\frac{x}{(\cos\theta)^n}+\underset{2}{A}\frac{x^2}{(\cos 2\theta)^n}+\&c.$$

$$\underset{1}{A}x\frac{\sin\theta}{(\cos\theta)^n}+\underset{2}{A}x^2\frac{\sin 2\theta}{(\cos 2\theta)^n}+\&c.$$

and

$$\underset{1}{A}\frac{x}{(\sin\theta)^n}+\underset{2}{A}\frac{x^2}{(\sin 2\theta)^n}+\&c.$$

$$\underset{1}{A}x\frac{\cos\theta}{(\sin\theta)^n}+\underset{2}{A}x^2\frac{\cos 2\theta}{(\sin 2\theta)^n}+\&c.$$

or else they depend partly on these and partly on other series, containing the powers of the sines or cosines of an arc in arithmetical progression in their numerators, which is a species whose sums are easily found. For the sake of brevity, I shall make use of the general term of any series with the characteristic S prefixed to it to denote that series. Beginning then with the series $\underset{i}{S} Ax^i \frac{(\sin i\theta)^m}{(\cos i\theta)^n}$, we observe that when m is an even number, we have

$$\underset{i}{S} Ax^i \frac{(\sin i\theta)^m}{(\cos i\theta)^n} = \underset{i}{S} Ax^i \frac{\left\{1-(\cos i\theta)^2\right\}^{\frac{m}{2}}}{(\cos i\theta)^n} = \underset{i}{S} Ax^i \frac{1}{(\cos i\theta)^n} - \frac{m}{2} \underset{i}{S} Ax^i \frac{1}{(\cos i\theta)^{n-2}}$$

$$+ \frac{m.\overline{m-2}}{2.4} \underset{i}{S} Ax^i \frac{1}{(\cos i\theta)^{n-4}} - \&c. \qquad (a)$$

this series will always terminate when m is an even number; and if m is greater than n, the last term will have no cosines in its denominator: if $m=n$, the last term will be $\underset{i}{S} Ax^i$; and if m is less than n, the last term will be $\underset{i}{S} Ax^i \frac{1}{(\cos i\theta)^{n-m}}$; so that in all cases when m is an even number, the series in question will depend on series of the form $\underset{i}{S} Ax^i \frac{1}{(\cos i\theta)^n}$, or on others whose sums are known.

Let us now consider the case of m = an odd number; then we have

$$\underset{i}{S} Ax^i \frac{(\sin i\theta)^m}{(\cos i\theta)^n} = \underset{i}{S} Ax^i \frac{\sin i\theta . \left\{1-(\cos i\theta)^2\right\}^{\frac{m-1}{2}}}{(\cos i\theta)^n} =$$

$$= \underset{i}{S} Ax^i \frac{(\sin i\theta}{(\cos i\theta)^n} - \frac{m-1}{2} \underset{i}{S} Ax^i \frac{\sin i\theta}{(\cos i\theta)^{n-2}} + \frac{\overline{m-1}.\overline{m-3}}{2.4} - \underset{i}{S} Ax^i \frac{\sin i\theta}{(\cos i\theta)^{n-4}} + \&c. (b)$$

This series always terminates when m is an odd number; and in a similar manner we shall find the two following:

$$\underset{i}{\mathrm{SA}}x^i \frac{(\cos i\theta)^m}{(\sin i\theta)^n} = \underset{i}{\mathrm{SA}}x^i \frac{\left\{1-(\sin i\theta)^2\right\}^{\frac{m}{2}}}{(\sin i\theta)^n} =$$

$$= \underset{i}{\mathrm{SA}}x^i \frac{1}{(\sin i\theta)^n} - \frac{m}{2}\underset{i}{\mathrm{SA}}x^i \frac{1}{(\sin i\theta)^{n-2}} + \frac{m \cdot m-2}{2 \cdot 4}\underset{i}{\mathrm{SA}}x^i \frac{1}{(\sin i\theta)^{n-4}} - \&c. \quad (c)$$

when m is an even number, and

$$\underset{i}{\mathrm{SA}}x^i \frac{(\cos i\theta)^m}{(\sin i\theta)^n} = \underset{i}{\mathrm{SA}}x^i \frac{\cos i\theta \left\{1-(\sin i\theta)^2\right\}^{\frac{m-1}{2}}}{(\sin i\theta)^n} =$$

$$= \underset{i}{\mathrm{SA}}x^i \frac{\cos i\theta}{(\sin i\theta)^n} - \frac{m-1}{2}\underset{i}{\mathrm{SA}}x^i \frac{\cos i\theta}{(\sin i\theta)^{n-2}} + \frac{m-1 \cdot m-3}{2 \cdot 4}\underset{i}{\mathrm{SA}}x^i \frac{\cos i\theta}{(\sin i\theta)^{n-4}} \&c. \quad (d)$$

when m is an odd number.

Let us now propose to investigate the sum of the series

$$\frac{\underset{1}{\mathrm{A}}x}{(\sin \theta)^n} + \frac{\underset{2}{\mathrm{A}}x^2}{(\sin 2\theta)^n} + \frac{\underset{3}{\mathrm{A}}x^3}{(\sin 3\theta)^n} \&c.$$

Assume $\psi x = \underset{1}{\mathrm{A}}x + \underset{2}{\mathrm{A}}x^2 + \underset{3}{\mathrm{A}}x^3 + \&c.$

Put v^{2z} for x; then it becomes

$$\psi v^{2z} = \underset{1}{\mathrm{A}}v^{2z} + \underset{2}{\mathrm{A}}v^{4z} + \underset{3}{\mathrm{A}}v^{6z} + \&c.$$

Integrate both sides, observing that $\Sigma v^{2iz} = \frac{v^{2iz}}{v^2-1}$; then we have

$$\Sigma\psi v^{2z} = \underset{1}{\mathrm{A}}\frac{v^{2z}}{v^2-1} + \underset{2}{\mathrm{A}}\frac{v^{4z}}{v^4-1} + \underset{3}{\mathrm{A}}\frac{v^{6z}}{v^6-1} + \&c.$$

Integrate again, and after the n^{th} integration we shall have

$$\Sigma^n \psi v^{2z} = \underset{1}{\mathrm{A}}\frac{v^{2z}}{(v^2-1)^n} + \underset{2}{\mathrm{A}}\frac{v^{4z}}{(v^4-1)^n} + \underset{3}{\mathrm{A}}\frac{v^{6z}}{(v^6-1)^n} + \&c.$$

Now let $v = \cos\theta \pm \sqrt{-1}\sin\theta$; then our equation becomes

$$(2\sqrt{-1})^n \Sigma^n \psi^{2z} = \underset{1}{\mathrm{A}}\frac{v^{2z-n}}{(\sin\theta)^n} + \underset{2}{\mathrm{A}}\frac{v^{4z-2n}}{(\sin 2\theta)^n} + \underset{3}{\mathrm{A}}\frac{v^{6z-3n}}{(\sin 3\theta)^n} + \&c.$$

Put $z + \frac{n}{2}$ for z, and we have

$$(2\sqrt{-1})^n \Sigma^n \psi v^{2z+n} = \underset{1}{\mathrm{A}}\frac{v^{2z}}{(\sin\theta)^n} + \underset{2}{\mathrm{A}}\frac{v^{4z}}{(\sin 2\theta)^n} + \underset{3}{\mathrm{A}}\frac{v^{6z}}{(\sin 3\theta)^n} + \&c.$$

$$= \underset{1}{A}\frac{x}{(\sin\theta)^n} + \underset{2}{A}\frac{x^2}{(\sin 2\theta)^n} + \underset{3}{A}\frac{x^3}{(\sin 3\theta)^n} + \&c. \qquad (1)$$

If after the integration we had put $\overset{-1}{v}$ instead of v, and then $v = \cos\theta \pm \sqrt{-1}\sin\theta$, we should have found

$$(-2\sqrt{-1})^n \Sigma^n \psi \overset{-2z-n}{v} = \underset{1}{A}\frac{\overset{-2x}{v}}{(\sin\theta)^n} + \underset{2}{A}\frac{\overset{-4x}{v}}{(\sin 2\theta)^n} + \underset{3}{A}\frac{\overset{-6x}{v}}{(\sin 3\theta)^n} + \&c.$$

$$= \underset{1}{A}\frac{\overset{-1}{x}}{(\sin\theta)^n} + \underset{2}{A}\frac{\overset{-2}{x}}{(\sin 2\theta)^n} + \underset{3}{A}\frac{\overset{-3}{x}}{(\sin 3\theta)^n} + \qquad (2)$$

Neither of the integrals exhibited in (1) and (2) are integrable in the most simple cases, and it is only by their combination that I have been able to obtain the sums of any series. Let us suppose $A = 1$, $\underset{2}{A} = -1$, $\underset{3}{A} = 1$, $\underset{4}{A} = -1$, &c. then

$$\psi \overset{2z+n}{v} = \frac{v^{2z+n}}{1+v^{2z+n}} \text{ and } \psi \overset{-2z-n}{v} = \frac{\overset{-2z-n}{v}}{1+\overset{-2z-n}{v}}; \text{ also let } n = 1;$$

then the difference of the two series is

$$(2\sqrt{-1})\Sigma\left\{\frac{v^{2z+1}}{1+v^{2z+1}} + \frac{\overset{-2z-1}{v}}{1+\overset{-2z-1}{v}}\right\} = \frac{x-\overset{-1}{x}}{\sin\theta} - \frac{x^2-\overset{-2}{x}}{\sin 2\theta} + \frac{x^3-\overset{-3}{x}}{\sin 3\theta} \&c.$$

but the integral on the left side of this equation becomes $(2\sqrt{-1})\Sigma(1)$ which is equal to $2\sqrt{-1}(z+b)$: hence, since $z = \frac{\log x}{2\log v}$ we have

$$2\sqrt{-1}\left\{\frac{\log x}{2\log v} + b\right\} = \frac{x-\overset{-1}{x}}{\sin\theta} - \frac{x^2-\overset{-2}{x}}{\sin 2\theta} + \frac{x^3-\overset{-3}{x}}{\sin 3\theta} + \&c.$$

If $x = 1$, $b = o$, and since $\log v = \theta\sqrt{-1}$, our series becomes

$$\frac{\log x}{\theta} = \frac{x-\overset{-1}{x}}{\sin\theta} - \frac{x^2-\overset{-2}{x}}{\sin 2\theta} + \frac{x^3-\overset{-3}{x}}{\sin 3\theta} - \&c. \qquad (3)$$

let $x = \cos\theta' \pm \sqrt{-1}, \sin\theta'$; then, since $\log x$ will become $\theta'\sqrt{-1}$, by dividing both sides by $2\sqrt{-1}$, we shall have

$$\frac{\theta'}{\theta} = \frac{\sin\theta'}{\sin\theta} - \frac{\sin 2\theta'}{\sin 2\theta} + \frac{\sin 3\theta'}{\sin 3\theta} - \&c. \qquad (4)$$

The series (3) is integrable when multiplied by $\frac{dx}{x}$, and this operation may be repeated any number of times; the first operation produces

$$\frac{(\log x)^2}{2\theta} \pm c_1 = \frac{x+x^{-1}}{1 \sin\theta} - \frac{x^2+x^{-2}}{2 \sin 2\theta} - \frac{x^3+x^{-3}}{3 \sin 3\theta} + \&c.$$

the value of the constant c_1, which is equal to the series

$$c_1 = 2\left\{\frac{1}{1 \sin\theta} - \frac{1}{2 \sin 2\theta} + \frac{1}{3 \sin 3\theta} - \&c.\right\}$$

cannot be determined from this equation, but by a second multiplication by $\frac{dx}{x}$ and again integrating, it may readily be found: this second operation gives

$$\frac{(\log x)^3}{2 \cdot 3 \cdot \theta} + \frac{\log x}{1} c_1 + c_2 = \frac{x-x^{-1}}{1^2 \sin\theta} - \frac{x^2-x^{-2}}{2^2 \sin 2\theta} + \frac{x^3-x^{-3}}{3^2 \sin 3\theta} - \&c.$$

If $x=1$, $c_2 = 0$, put $x = \cos\theta + \sqrt{-1}\sin\theta$, then we have

$$\frac{(\theta\sqrt{-1})^3}{2 \cdot 3 \cdot \theta} + \frac{\theta\sqrt{-1}}{1} c_1 = 2\sqrt{-1}\left\{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \&c.\right\} = 2\sqrt{-1}\, S\frac{\pm 1}{i^2}\text{*}$$

From this equation the value of c_1 may be found; it is

$$c_1 = \frac{1}{\theta}\left\{\frac{\theta^2}{6} + 2\, S\frac{\pm 1}{i^2}\right\}$$

The value of c_1 thus found, we have the series

$$\frac{(\log x)^2}{2 \cdot \theta} + \frac{1}{\theta}\left\{\frac{\theta^2}{6} + 2S\frac{\pm 1}{i^2}\right\} = \frac{x+x^{-1}}{1 \sin\theta} - \frac{x^2+x^{-2}}{2 \sin 2\theta} + \frac{x^3+x^{-3}}{3 \sin 3\theta} - \&c. \quad (5)$$

and

$$\frac{(\log x)^3}{1.2.3.\theta} + \frac{\log x}{\theta}\left\{\frac{\theta^2}{6} + 2S\frac{\pm 1}{i^2}\right\} = \frac{x-x^{-1}}{1^2 \sin\theta} - \frac{x^2-x^{-2}}{2^2 \sin 2\theta} + \frac{x^3-x^{-3}}{3^2 \sin 3\theta} - \&c. \quad (6)$$

In the first of these put $x = \cos\theta + \sqrt{-1}\sin\theta$, and it becomes

$$-\frac{\theta}{6} + \frac{1}{\theta} S\frac{\pm 1}{i^2} = \frac{\cot\theta}{1} - \frac{\cot 2\theta}{2} + \frac{\cot 3\theta}{3} - \&c. \quad (7)$$

* Throughout the course of this Paper I shall have continual occasion to employ the series $\frac{1}{1^{2n}} - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \&c.$; they can always be expressed by means of the numbers of BERNOUILLI, and the powers of π, and for the sake of brevity I shall always denote them by $S\frac{\pm 1}{i^{2n}}$.

By continuing to multiply (3) by $\frac{dx}{x}$ and integrating, it is easy to perceive that we should arrive at the two following theorems—

$$\frac{(\log x)^{2k}}{1.2..2k.\theta} + \frac{(\log x)^{2k-2}}{1.2...2k-2}c_1 + \&c. + c_{2k-1} =$$

$$\frac{x+x^{-1}}{1^{2k-1}\sin\theta} - \frac{x^2+x^{-2}}{2^{2k-1}\sin 2\theta} + \frac{x^3+x^{-3}}{3^{2k-1}\sin 3\theta} - \&c. \qquad (8)$$

$$\frac{(\log x)^{2k+1}}{1.2..2k+1.\theta} + \frac{(\log x)^{2k-1}}{1.2..2k-1}c_1 + \&c. + \frac{\log x}{1}c_{2k-1} =$$

$$\frac{x-x^{-1}}{1^{2k}\sin\theta} - \frac{x^2-x^{-2}}{2^{2k}\sin 2\theta} + \frac{x^3-x^{-3}}{3^{2k}\sin 3\theta} - \&c.$$

the constants c_1, c_3, c_5, &c. may easily be determined from each other, the value of c_1 has been already found, that of c_3 and c_5 are as follows:

$$c_3 = \frac{2}{\theta}S\frac{\pm 1}{i^4} + \frac{2\theta}{6}S\frac{\mp 1}{i^2} + \frac{7\theta^3}{360}$$

$$c_5 = \frac{2}{\theta}S\frac{\pm 1}{i^6} + \frac{2\theta}{6}S\frac{\pm 1}{i^4} + 2\,\frac{7\theta^3}{360}S\frac{\pm 1}{i^2} + \theta^5\frac{31}{5040}$$

a variety of series are deducible from those of (8); I shall only mention two of them;

$$\frac{\cot\theta}{1^{2k-1}} - \frac{\cot 2\theta}{2^{2k-1}} + \frac{\cot 3\theta}{3^{2k-1}} - \&c.$$

and $$\frac{1}{1^{2k}}\cdot\frac{\sin\theta'}{\sin\theta} - \frac{1}{2^{2k}}\cdot\frac{\sin 2\theta'}{\sin 2\theta} + \frac{1}{3^{2k}}\cdot\frac{\sin 3\theta'}{\sin 3\theta} - \&c.$$

Returning to the formulæ (1) and (2) by addition and subtraction, we shall have when n is even

$$(2\sqrt{-1})^n\Sigma^n\left\{\psi v^{2z+n} + (-1)^n\psi v^{-2z-n}\right\} =$$

$$A_1\frac{x+x^{-1}}{(\sin\theta)^n} + A_2\frac{x^2+x^{-2}}{(\sin 2\theta)^n} + A_3\frac{x^3+x^{-3}}{(\sin 3\theta)^n} + \&c.$$

and when n is an odd number,

$$(2\sqrt{-1})^n \Sigma^n \left\{ \psi v^{2z+n} + (-1)^n \psi v^{-2z-n} \right\} =$$
$$= \underset{1}{A} \frac{x - x^{-1}}{(\sin\theta)^n} + \underset{2}{A} \frac{x^2 - x^{-2}}{(\sin 2\theta)^n} + \underset{3}{A} \frac{x^3 - x^{-3}}{(\sin 3\theta)^n} + \&c.$$

From these expressions it appears, that the reason why we have succeeded in the integrations is, because we had so assumed ψ, that the sum, $\psi v^{2z+n} + \psi v^{-2z-n}$ is a constant quantity; the same success must follow whenever this condition is fulfilled: and hence, we have a method of discovering the sums of a great variety of series, containing the powers of the sines of arcs in arithmetical progression in their denominators, by solving the functional equation $\psi v^{2z+n} + \psi v^{-2z-n} = c$. This is fortunately one of a class whose general solution I have arrived at;* it is

$$\psi v^{2z+n} = \frac{c\phi v^{2z+n}}{\phi v^{2z+n} + \phi v^{-2z-n}} \text{ or } \psi x = \frac{c\phi x}{\phi x + \phi \frac{1}{x}}$$

In the example I have employed n was supposed equal to unity; if this is not the case, we should have found

$$(2\sqrt{-1})^n \Sigma^n(1) = \frac{x \pm x^{-1}}{(\sin\theta)^n} - \frac{x^2 \pm x^{-2}}{(\sin 2\theta)^n} + \frac{x^3 \pm x^{-3}}{(\sin 3\theta)^n} - \&c.$$

If in the functional equation we put $c = 1$, and $\phi x = \tan^{-1} x$, then we have $\psi x = \frac{2}{\pi} \tan^{-1} x$, and

$$(2\sqrt{-1})^n \Sigma^n(1) = \frac{2}{\pi} \left\{ \frac{x \pm x^{-1}}{1\,(\sin\theta)^n} - \frac{x^3 \pm x^{-3}}{3\,(\sin 3\theta)^n} + \frac{x^5 \pm x^{-5}}{5\,(\sin 5\theta)^n} - \&c. \right\}$$

the upper or under sign being used as n, is even or odd; if $n = 1$, the constant is zero; and we have

$$\frac{\pi \log x}{2\theta} = \frac{x - x^{-1}}{1 \sin\theta} - \frac{x^3 - x^{-3}}{3 \sin 3\theta} + \frac{x^5 - x^{-5}}{5 \sin 5\theta} - \&c. \qquad (9)$$

* See Philosophical Transactions for 1817, p. 202.

this may be multiplied by $\frac{dx}{x}$, and integrated any number of times in the same manner as (3), and the results would be

$$\frac{\pi}{2\theta}\cdot\frac{(\log x)^{2k}}{1.2...2k}+\frac{(\log x)^{2k-2}}{1.2..2k-2}\,c_{1}+\&c.+c_{2k-1}=$$

$$=\frac{x+x^{-1}}{1^{2k}\sin\theta}-\frac{x^{3}+x^{-3}}{3^{2k}\sin 3\theta}+\frac{x^{5}+x^{-5}}{5^{2k}\sin 5\theta}-\&c. \qquad (1,1)$$

$$\frac{\pi}{2\theta}\cdot\frac{(\log x)^{2k+1}}{1.2..2k+1}+\frac{(\log x)^{2k-1}}{1.2..2k-1}\,c_{1}+\&c.+\frac{\log x}{1}\,c_{2k-1}=$$

$$=\frac{x-x^{-1}}{1^{2k+1}\sin\theta}-\frac{x^{3}-x^{-3}}{3^{2k+1}\sin 3\theta}+\frac{x^{5}-x^{-5}}{5^{2k+1}\sin 5\theta}-\&c.$$

and these constants may be determined one from the other in the same manner as the former. I shall only give the value of the first, in order to compare the value of the series to which it is equal, with the sum of the same series deduced in another manner.

$$c=\frac{\pi\theta}{12}+\frac{2}{\theta}\,S\,\frac{\pm 1}{\overline{2i+1}^{3}}=2\left\{\frac{1}{1^{2}\sin\theta}-\frac{1}{3^{2}\sin 3\theta}+\frac{1}{5^{2}\sin 5\theta}-\&c.\right\}(1,2)$$

In order to ascertain the sums of series which contain cosines in their denominators, we must use an artifice which I shall now explain.

Assuming as before $\psi x=A_{1}x+A_{2}x^{2}+A_{3}x^{3}+\&c$, and putting v^{2z} for x, we have

$$\psi v^{2z}=A_{1}v^{2z}+A_{2}v^{4z}+A_{3}v^{6z}+\&c.$$

If we were now to integrate this, we should introduce into the denominator of any term $v^{2i}-1$; but we want to introduce the same expression with the signs of both terms positive. If we multiply both sides by $(-1)^{z}$ and then integrate, we shall have first

$$\overline{(-1)}^{z}\,\psi v^{2z}=A_{1}v^{2z}\,\overline{(-1)}^{z}+A_{2}v^{4z}\,\overline{(-1)}^{z}+A_{3}v^{6z}\,\overline{(-1)}^{z}+\&c.$$

And since $\Delta\,(-1)^{\overline{x}}\,v^{2ix} = -(-1)^{\overline{x}}\,v^{2ix+2i} - (-1)^{\overline{x}}\,v^{2ix} = -(v^{2i}+1)\,(-1)^{x}\,v^{2ix}$ we find

$$\Sigma\,(-1)^{x}\,v^{2ix} = -\frac{v^{2ix}}{v^{2i}+1}$$

Integrating each term separately, we have

$$\Sigma(-1)^{\overline{x}}\,\psi v^{2x} = -\left\{ \underset{1}{A}\frac{v^{2x}(-1)^{\overline{x}}}{v^{2}+1} + \underset{2}{A}\frac{v^{4x}(-1)^{\overline{x}}}{v^{4}+1} + \underset{3}{A}\frac{v^{6x}(-1)^{\overline{x}}}{v^{6}+1} + \&c. \right\}$$

Let this integration be repeated n times, it will give

$$(-1)^{n}\Sigma^{n}(-1)^{\overline{x}}\,\psi v^{2x} = \underset{1}{A}\frac{v^{2x}(-1)^{\overline{x}}}{(v^{2}+1)^{n}} + \underset{2}{A}\frac{v^{4x}(-1)^{\overline{x}}}{(v^{4}+1)^{n}} + \underset{3}{A}\frac{v^{6x}(-1)^{\overline{x}}}{(v^{6}+1)^{n}} + \&c.$$

Let $v = \cos\theta\sqrt{-1}\,\sin\theta$, and $x + \frac{n}{2}$ for x; this becomes

$$(-2)^{n}\Sigma^{n}(-1)^{\overline{x}-\frac{n}{2}}\,\psi v^{2x+n} = (-1)^{\overline{x}-\frac{n}{2}}\left\{ \underset{1}{A}\frac{v^{2x}}{(\cos\theta)^{n}} + \underset{2}{A}\frac{v^{4x}}{(\cos 2\theta)^{n}} + \underset{3}{A}\frac{v^{6x}}{(\cos 3\theta)^{n}} + \&c. \right\}$$

And finally,

$$(-2)^{n}(-1)^{x}\,\Sigma^{n}(-1)^{\overline{x}}\,\psi v^{2x+n} = \underset{1}{A}\frac{v^{2x}}{(\cos\theta)^{n}} + \underset{2}{A}\frac{v^{4x}}{(\cos 2\theta)^{n}} + \underset{3}{A}\frac{v^{6x}}{(\cos 3\theta)^{n}} + \&c.$$

$$= \underset{1}{A}\frac{x}{(\cos\theta)^{n}} + \underset{2}{A}\frac{x^{2}}{(\cos 2\theta)^{n}} + \underset{3}{A}\frac{x^{3}}{(\cos 3\theta)^{n}} + \&c. \qquad (1,3)$$

The integrations here indicated will, as in a former instance, generally surpass the powers of analysis in its present state; but a contrivance similar to that which has been already stated, will in many cases elude the difficulty: the artifice consists in investigating another similar series arranged according to the descending powers of the variable, integrating it in the same manner as we have that marked (1,3), and adding these two results, we shall in many cases have a function which is integrable, and the two series become equal in the case of $x = 1$. By commencing with the descending series $\psi v^{-2z} = \underset{1}{A} v^{-2z} + \underset{2}{A} v^{-4z} + \underset{3}{A} v^{-6z} + \&c.$ multiplying by $(-1)^{-z}$ and integrating, we shall get the expression

$$(-2)^n (-1)^z \Sigma^n (-1)^{-z} \psi v^{-2z-n} = \underset{1}{A} \frac{v^{-2z}}{(\cos\theta)^n} + \underset{2}{A} \frac{v^{-4z}}{(\cos 2\theta)^n} + \underset{3}{A} \frac{v^{-6z}}{(\cos. 3\theta)^n} - \&c.$$

$$= \underset{1}{A} \frac{x^{-1}}{(\cos\theta)^n} + \underset{2}{A} \frac{x^{-2}}{(\cos 2\theta)^n} + \underset{3}{A} \frac{x^{-3}}{(\cos 3\theta)^n} + \&c. \qquad (1,4)$$

There occur very few cases in which it is possible to execute the integrations in (1,3) and (1,4); by adding the two together, we have

$$(-2)^n (-1)^z \Sigma^n (-1)^{-z} \left\{ \psi v^{2z+n} + \psi v^{-2z-n} \right\} = \underset{1}{A} \frac{x + x^{-1}}{(\cos\theta)^n} + \underset{2}{A} \frac{x^2 + x^{-2}}{(\cos 2\theta)^n}$$

$$+ \underset{3}{A} \frac{x^3 + x^{-3}}{(\cos 3\theta)^n} + \&c. \qquad (1,5)$$

Here we may observe that the new series is exactly double either of the others (1,3) or (1,4) when $x = 1$; also, that the integration on the left side can be executed any number of times, whenever $\psi v^{2x+n}\ \psi v^{-2z-n}$ is a constant quantity; the forms of the function ψ, which fulfil this condition, have already been given. Let $\psi x = \frac{x}{1+x}$, then $\underset{1}{A} = 1$, $\underset{2}{A} = -1$, $\underset{3}{A} = 1$, &c. and since $\psi v^{2z+n} + \psi v^{-2z-n} = 1$, we have

$$(-2)^n (-1)^z \Sigma^n (-1)^{-z} = \frac{x + x^{-1}}{(\cos\theta)^n} - \frac{x^2 + x^{-2}}{(\cos 2\theta)^n} + \frac{x^3 + x^{-3}}{(\cos 3\theta)^n} + \&c.$$

These integrations are easily executed; and commencing with $n = 1$, we have

$$-2 \,.\, (-1)^z \frac{(-1)^{-z}}{-1-1} + 2b(-1)^z = 1 + 2b(-1)^z = \frac{x + x^{-1}}{\cos\theta} - \frac{x^2 + x^{-2}}{\cos 2\theta} + \&c$$

In order to determine the constant b, put $x = \cos\theta + \sqrt{-1}\ \sin\theta$; then, since z in that case becomes $\frac{1}{2}$, we have

$$1 + 2b\sqrt{-1} = 2 - 2 + 2 - 2 + \&c. = 1$$

hence $b = o$, and we have

$$1 = \frac{x + x^{-1}}{\cos\theta} - \frac{x^2 + x^{-2}}{\cos 2\theta} + \frac{x^3 + x^{-3}}{\cos 3\theta} - \text{\&c.} \qquad (1,6)$$

Continuing to integrate, it will be found that all the constants are zero, and we shall arrive at the following theorem;

$$1 = \frac{x + x^{-1}}{(\cos\theta)^n} - \frac{x + x^{-2}}{(\cos 2\theta)^n} + \frac{x^3 + x^{-3}}{(\cos 3\theta)^n} - \text{\&c.} \qquad (1,7)$$

Let $x = -x$, then it becomes

$$-1 = \frac{x + x^{-1}}{(\cos\theta)^n} + \frac{x^2 + x^{-2}}{(\cos 2\theta)^n} + \frac{x^3 + x^{-3}}{(\cos 3\theta)^n} + \text{\&c.} \qquad (1,8)$$

Putting $x = 1$ in both these, we have

$$\frac{1}{2} = \frac{1}{(\cos\theta)^n} - \frac{1}{(\cos 2\theta)^n} + \frac{1}{(\cos 3\theta)^n} \text{\&c.} \qquad (1,9)$$

$$-\frac{1}{2} = \frac{1}{(\cos\theta)^n} + \frac{1}{(\cos 2\theta)^n} + \frac{1}{(\cos 3\theta)^n} + \text{\&c.} \qquad (2,1)$$

I propose in the next place to determine the value of the series

$$\frac{1}{1^{2k}(\cos\theta)^n} - \frac{1}{2^{2k}(\cos 2\theta)^n} + \frac{1}{3^{2k}(\cos 3\theta)^n} - \text{\&c.}$$

This may be accomplished by multiplying (1,7) by $\frac{dx}{x}$, and integrating; this operation, being performed on it 2^k times, will produce the series whose sum is required; the first integration gives

$$\frac{\log x}{1} + c = \frac{x - x^{-1}}{1(\cos\theta)^n} - \frac{x^2 - x^{-2}}{2(\cos 2\theta)^n} + \frac{x^3 - x^{-3}}{3(\cos 3\theta)^n} - \text{\&c.}$$

If $x = 1$, $c = o$, the second operation gives

$$\frac{(\log x)^2}{1 \cdot 2} + \underset{2}{c} = \frac{x + x^{-1}}{1^2(\cos\theta)^n} - \frac{x^2 + x^{-2}}{2^2(\cos 2\theta)^n} - \text{\&c.}$$

If $x = 1$ $\qquad c_{2,n} = \frac{2}{1^2(\cos\theta)^n} - \frac{2}{2^2(\cos 2\theta)^n} + \text{\&c.}$

In order to determine $c_{2,n}$ put $x = \cos\theta + \sqrt{-1}\sin\theta$, then we have

$$-\frac{\theta^2}{2}+c_{2,n}=\frac{2}{1^2(\cos\theta)^{n-1}}-\frac{2}{2^2(\cos\theta)^{n-1}}+\&c.=c_{2,n-1}.$$

The equation $c_{2,n}-c_{2,n-1}=\frac{\theta^2}{2}$ being integrated, gives

$$c_{2,n}=\frac{n}{2}\frac{\theta^2}{2}+b$$

If $n=0$ $\quad c_{2,0}=b=\frac{2}{1^2}-\frac{2}{2^2}+\frac{2}{3^2}-\&c.=2S\frac{\pm 1}{i^2}$

Hence $\quad c_{2,n}=\frac{n}{1}\cdot\frac{\theta^2}{2}+2S\frac{\pm 1}{i^2}$ and

$$\frac{(\log.x)^2}{1.2}+\frac{n}{1}\cdot\frac{\theta^2}{2}+2S\frac{\pm 1}{i^2}=\frac{x+x^{-1}}{1^2(\cos\theta)^n}-\frac{x^2+x^{-2}}{2^2(\cos 2\theta)^n}+\frac{x^3+x^{-3}}{3^2(\cos 3\theta)^n}\&c. \quad (2,2)$$

These integrations being repeated, we shall arrive at the two following expressions:

$$\frac{(\log x)^{2k}}{1.2\ldots 2k}+\frac{(\log x)^{2k-2}}{1.2\ldots 2k-2}c_{2,n}+\&c.+\frac{(\log x)^2}{1.2}c_{2k-2,n}+c_{2k,n}=$$

$$=\frac{x+x^{-1}}{1^{2k}(\cos\theta)^n}-\frac{x^2+x^{-2}}{2^{2k}(\cos 2\theta)^n}+\frac{x^3+x^{-3}}{3^{2k}(\cos 3\theta)^n}+\&c. \qquad (2,3)$$

$$\frac{(\log x)^{2k+1}}{1.2\ldots 2k+1}+\frac{(\log x)^{2k-1}}{1.2\ldots 2k-1}c_{2,n}+\&c.\,c_{2k,n}=$$

$$\frac{x-x^{-1}}{1^{2k+1}(\cos\theta)^n}-\frac{x^2-x^{-2}}{2^{2k+1}(\cos 2\theta)^n}+\frac{x^3-x^{-3}}{3^{2k+1}(\cos\theta)^n}-\&c.$$

It now becomes necessary to determine the value of $c_{2k,n}$, which is equal to twice the sum of the series we are investigating; for if $x=1$

$$c_{2k,n}=\frac{2}{1^{2k}(\cos\theta)^n}-\frac{2}{2^{2k}(\cos 2\theta)^n}+\frac{2}{3^{2k}(\cos 3\theta)^n}-\&c.$$

For this purpose put in the first of the equations (2,3) $x=\cos\theta+\sqrt{-1}\sin\theta$, then the series on the right hand is equal to $c_{2k,n-1}$, and we have for determining $c_{2k,n}$ the equation of finite differences.

$$\frac{(\theta\sqrt{-1})^{2k}}{1.2\ldots 2k}+\frac{(\theta\sqrt{-1})^{2k-2}}{1.2\ldots 2k-2}c_{2,n}+\frac{(\theta\sqrt{-1})^{2k-4}}{1.2\ldots 2k-4}c_{4,n}+\&c.+c_{2k,n}=c_{2k,n-1}$$

In order to integrate this equation, let us suppose $c_{2k,n}$ to represent the co-efficient of r^{2k} in the developement of $\frac{f(r)}{(\alpha r)^n}$

where $\quad \alpha(r) = 1 + Ar^2 + Br^4 + Cr^6 + \&c.$

then $\quad \frac{f(r)}{(\alpha r)^n} = 1 + rc^2_{2,n} + r^4 c_{4,n} + r^6 c_{6,n} + \&c.$

If this be multiplied by αr, it becomes

$$\begin{aligned}\frac{f(r)}{(\alpha r)^{n-1}} = 1 + r^2 c_{2,n} &+ r^4 c_{4,n} + r^6 c_{6,n} + \\ r^2 A &+ r^4 A c_{2,n} + r^6 A c_{4,n} \\ &\quad r^4 B + r^6 B c_{2,n} \\ &\qquad + r^6 C\end{aligned}$$

But the co-efficient of r^{2k} in this series is equal to $c_{2k,n-1}$ hence

$$c_{2k,n} + Ac_{2k-2,n} + Bc_{2k-4,n} + \&c. = c_{2k,n-1}$$

This equation will become the one in question, if we make

$$A = \frac{(\theta\sqrt{-1})^2}{1.2} B = \frac{(\theta\sqrt{-1})^4}{1.2.3.4} C = \frac{(\theta\sqrt{-1})^6}{1.2..6} \&c.$$

This produces

$$c_{2k,n} + \frac{(\theta\sqrt{-1})^2}{1.2} c_{2k-2,n} + \frac{(\theta\sqrt{-1})^4}{1.2.3.4} c_{2k-4,n} + \&c. = c_{2k,n-1}$$

We have now only to determine the form of $f(r)$, and this may be easily accomplished since the values of $c_{2k,o}$ are known; for if we put $n = o$

$$\begin{aligned} f(r) &= 1 + r^2 c_{2,o} + r^4 c_{4,o} + r^6 c_{6,o} + \&c. \\ &= 1 + 2r^2 S\frac{\pm 1}{i^2} + 2r^4 S\frac{\pm 1}{i^4} + 2r^6 S\frac{\pm 1}{i^6} + \&c.\end{aligned}$$

Therefore $c_{2k,n}$ is equal to the co-efficient of r^{2k} in the developement of

$$\frac{1 + 2r^2 S\frac{\pm 1}{i^2} + 2r^4 S\frac{\pm 1}{i^4} + \&c.}{(\cos r\theta)^n}$$

Or if $(\cos r\theta)^{-n} = 1 + A'_n \theta^2 r^2 + B'_n \theta^4 r^3 + C'_n \theta^6 r^6 + \&c.$ then

$$c_{2k,n} = 2S\frac{\pm 1}{i^{2k}} + 2A'_n\theta^2 S\frac{\pm 1}{i^{2k-2}} + 2B'_n\theta^4 S\frac{\pm 1}{i^{2k-4}} + 2C'_n\theta^6 S\frac{\pm 1}{i^{2k-6}} + \&c.$$

The quantity $c_{2k,n}$ may now be considered as completely determined, since it only depends on the co-efficients of $(\cos\theta)^{-n}$, and the series marked by $S\frac{\pm 1}{i^{2k}}$, both which quantities are known; the latter being given by the powers of π and numbers of BERNOUILLI, whilst the values of the former in functions of n are given by LEGENDRE, in his Exercise de Calcul Integral, vol. iii. art. 149, 155.

In (2,3) let $x = 1$, and we have

$$\tfrac{1}{2}c_{2k,n} = \frac{1}{1^{2k}(\cos\theta)^n} - \frac{1}{2^{2k}(\cos 2\theta)^n} + \frac{1}{3^{2k}(\cos 3\theta)^n} - \&c. \qquad (2,4)$$

And if we put $x = v$ in the other series, it becomes

$$\frac{\theta}{2}\left\{\frac{(-\theta^2)^k}{1.2..2k+1} + \frac{(-\theta^2)^{k-1}}{1.2..2k-1}c_{2,n} + \&cc. + \frac{(-\theta^2)}{1.2.3}c_{2k-2,n} + \tfrac{1}{1}c_{2k,n}\right\} =$$

$$= \frac{\sin\theta}{1^{2k+1}(\cos\theta)^n} - \frac{\sin 2\theta}{2^{2k+1}(\cos 2\theta)^n} + \frac{\sin 3\theta}{3^{2k+1}(\cos 3\theta)^n} - \&c. \quad (2,5)$$

If $n = 1$ this series becomes

$$\frac{\tan\theta}{1^{2k+1}} - \frac{\tan 2\theta}{2^{2k+1}} + \frac{\tan 3\theta}{3^{2k+1}} - \&c. \qquad (2,6)$$

The series (2,4) may be changed into another, which contains sines both in the numerator and the denominator, for it is equal to

$$\tfrac{1}{2}c_{2k,n} = \frac{1}{1^{2k}}\left(\frac{\sin\theta}{\sin\theta.\cos\theta}\right)^n - \frac{1}{2^{2k}}\left(\frac{\sin 2\theta}{\sin 2\theta.\cos 2\theta}\right)^n + \frac{1}{3^{2k}}\left(\frac{\sin 3\theta}{\sin 3\theta.\cos 3\theta}\right)^n - \&c.$$

But this becomes, since $\sin\theta.\cos\theta = \tfrac{1}{2}\sin 2\theta$

$$\frac{1}{2^{n+1}}c_{2k,n} = \frac{1}{1^{2k}}\left(\frac{\sin\theta}{\sin 2\theta}\right)^n - \frac{1}{2^{2k}}\left(\frac{\sin 2\theta}{\sin 4\theta}\right)^n + \frac{1}{3^{2k}}\left(\frac{\sin 3\theta}{\sin 6\theta}\right)^n - \&c. \qquad (2,7)$$

By applying the theorems (a), (b), (c), and (d) to the series whose sums we have now investigated, we shall arrive at the value of many others which contain the powers of tangents and co-tangents in arithmetical progression, thus (1,9) combined with (*a*) will produce

$$\tfrac{1}{2}\left\{1-\frac{k}{1}+\frac{k.k-1}{1.2}-\&c.\right\}=\frac{(\sin\theta)^{2k}}{(\cos\theta)^{n}}-\frac{(\sin 2\theta)^{2k}}{(\cos 2\theta)^{n}}+\frac{(\sin 3\theta)^{2k}}{(\cos 3\theta)^{n}}-\&c.$$

But the left side of this equation is equal to $\frac{1}{2}(1-1)^{k}=0$. Hence

$$0=\frac{(\sin\theta)^{2k}}{(\cos\theta)^{n}}-\frac{(\sin 2\theta)^{2k}}{(\cos 2\theta)^{n}}+\frac{(\sin 3\theta)^{2k}}{(\cos 3\theta)^{n}}-\&c. \qquad (2,8)$$

And if $n=2k$, this produces a series of tangents

$$0=(\tan\theta)^{2k}-(\tan 2\theta)^{2k}+(\tan 3\theta)^{2k}-\&c. \qquad (2,9)$$

By means of the theorems already referred to, we may introduce into the numerators of each term of the series (2,4) the even powers of the sines of the same arcs whose co-sines occur in the denominator: putting $l=\frac{m}{2}$, we shall have

$$\tfrac{1}{2}\left\{c_{2k,n}-\frac{l}{1}c_{2k,n-2}+\frac{l.l-1}{1.2}c_{2k,n-4}-\&c.\right\}=$$
$$=\frac{(\sin\theta)^{2l}}{1^{2k}(\cos\theta)^{n}}-\frac{(\sin 2\theta)^{2l}}{2^{2k}(\cos 2\theta)^{n}}+\frac{(\sin 3\theta)^{2l}}{3^{2k}(\cos 3\theta)^{n}}-\&c. \qquad (3,1)$$

And if $n=2l$, this becomes

$$\tfrac{1}{2}\left\{c_{2k,2l}-\frac{l}{1}c_{2k,2l-2}+\frac{l.l-1}{1.2}c_{2k,2l-4}-\&c.\right\}=$$
$$=\frac{(\tan\theta)^{2l}}{1^{2k}}-\frac{(\tan 2\theta)^{2l}}{2^{2k}}+\frac{(\tan 3\theta)^{2l}}{3^{2k}}-\&c. \qquad (3,2)$$

If we call the sum of the series (2,5) $A_{k,n}$, and if we apply to it the formula (*b*), we shall have

$$A_{k,n}-\frac{l}{1}A_{k,n-2}+\frac{l.l-1}{1.2}A_{k,n-4}-\&c.=$$
$$=\frac{(\sin\theta)^{2l+1}}{1^{2k+1}(\cos\theta)^{n}}-\frac{(\sin 2\theta)^{2l+1}}{2^{2k+1}(\cos 2\theta)^{n}}+\frac{(\sin 3\theta)^{2l+1}}{3^{2k+1}(\cos 3\theta)^{n}}-\&c. \qquad (3,3)$$

And if $n=2l+1$, this becomes

$$A_{k,2l+1}-\frac{l}{1}A_{k,2l-1}+\&c.=$$
$$=\frac{(\tan\theta)^{2l+1}}{1^{2k+1}}-\frac{(\tan 2\theta)^{2l+1}}{2^{2k+1}}+\frac{(\tan 3\theta)^{2l+1}}{3^{2k+1}}-\&c. \qquad (3,4)$$

In the equation (1,4) if we make $\psi x = \tan^{-1} x$, we shall find

$$(-2)^n (-1)^z \Sigma^n (-1)^{-z} \left\{ \tan^{-1} v^{2z+n} + \tan^{-1} v^{-2z-n} \right\} =$$

$$= (-2)^n (-1)^z \Sigma^n (-1)^{-z} \cdot \frac{\pi}{4} = \frac{x+x^{-1}}{1(\cos\theta)^n} - \frac{x^3+x^{-3}}{3(\cos 3\theta)^n} + \frac{x^5+x^{-5}}{5(\cos 5\theta)^n} - \&c.$$

If the integrations here indicated are performed, it will be found that all the constants vanish, and ultimately that

$$\frac{\pi}{2} = \frac{x+x^{-1}}{1(\cos\theta)^n} - \frac{x^3+x^{-3}}{3(\cos 3\theta)^n} + \frac{x^5+x^{-5}}{5(\cos 5\theta)^n} - \&c. \qquad (3,5)$$

If $x = 1$

$$\frac{\pi}{4} = \frac{1}{1(\cos\theta)^n} - \frac{1}{3(\cos 3\theta)^n} + \frac{1}{5(\cos 5\theta)^n} - \&c. \qquad (3,6)$$

If we multiply (3,5) by $\frac{dx}{x}$ and integrate, we have

$$\frac{\pi}{2}\log x + C = \frac{x-x^{-1}}{1^2(\cos\theta)^n} - \frac{x^3-x^{-3}}{3^2(\cos 3\theta)^n} + \frac{x^5-x^{-5}}{5^2(\cos 5\theta)^n} - \&c. \quad (3,7)$$

If $x = 1$ $C = o$, let $x = \cos\theta' + \sqrt{-1}\sin\theta'$, then we have

$$\frac{\pi\theta'}{2} = \frac{\sin\theta'}{1^2(\cos\theta)^n} - \frac{\sin 3\theta'}{3(\cos 3\theta)^n} + \frac{\sin 5\theta'}{5(\cos 5\theta)^n} - \&c. \qquad (3,8)$$

The equation (3,7) may be multiplied any number of times by $\frac{dx}{x}$, and integrated; and the constants thus introduced may be determined in the same manner as those of the equations (2,3); these operations will give the values of series of the following form:

$$\frac{x+x^{-1}}{1^{2k+1}(\cos\theta)^n} - \frac{x^3+x^{-3}}{3^{2k+1}(\cos 3\theta)^n} + \frac{x^5+x^{-5}}{5^{2k+1}(\cos 5\theta)^n} - \&c.$$

$$\frac{x-x^{-1}}{1^{2k}(\cos\theta)^n} - \frac{x^3-x^{-1}}{3^{2k}(\cos 3\theta)} + \frac{x^5-x^{-5}}{5^{2k}\cos 5\theta} - \&c.$$

Numerous other series might be found by satisfying the equation $\psi x + \psi\frac{1}{x} = 1$, whose sums would be given by this process; and if instead of putting $\frac{1}{x}$ for x, we had substituted αx for x, where the function α is determined by the equation $\alpha^2 x = x$, many others would be discovered. This artifice is only a particular case of a much more general principle, which is of use in discovering certain values of the variable, in which a series admits of summation, but which generally is not expressible in finite terms. The principle is as follows: let K denote any operation, such as integration, either with respect to differential or finite differences, or any other operation, provided only $K(X+Y) = KX + KY$ Now let

$$K\psi x = A_1 x + A_2 x^2 + A_3 x^3 +$$

Put $\alpha x, \alpha^2 x, \ldots \alpha^{n-1} x$ for x, and the results will be

$$K\psi\alpha x = A_1 \alpha x + A_2 \alpha x^2 + A_3 \alpha x^3 +$$

$$K\psi\alpha^2 x = A_1 \alpha^2 x + A_2 \alpha^2 x^2 + A_3 \alpha^3 x^3 +$$

&c. &c.

$$K\psi\alpha^{n-1} x = A_1 \alpha^{n-1} x + A_2 \alpha^{n-1} x^2 + A_3 \alpha^{n-1} x^3 +$$

By adding all these together, we shall have $K\left\{\psi x + \psi\alpha x + \psi\alpha^2 x + \&c.\ \psi\alpha^{n-1} x\right\}$ equal to a series whose general term is $A_n\left\{x + \alpha x + \alpha^2 x + \&c. + \alpha^{n-1} x\right\}$. Now supposing we cannot perform the operation denoted by K on the function ψx, yet if ψ is of such a form that $\psi x + \psi\alpha x + \&c. + \psi\alpha^{n-1} x$ is equal to a function on which the operation K can be executed, then calling this new function ψ_1, we shall have

$$K\psi_1 x = SA_i\left\{x^i + \alpha x^i + \ldots \alpha^{n-1} x^i\right\}$$

And if α is such a function that $\alpha^n x = x$, a great variety of forms for ψ may be found, which will satisfy that condition. Now let x be determined by the equation $x = \alpha x$, and r being any root of this, we have $r = \alpha r = \alpha^2 r = \& = \alpha^{n-1} r$.

Consequently our equation becomes

$$K\psi_1 x = nSA_i r^i = n\left\{A_1 r + A_2 r^2 + A_3 r^3 + A_4 r^4 + \&c.\right\}$$

provided we put r for x after the operation K is executed; that is, we have found the values of the series

$$A_1 x + A_2 x^2 + A_3 x^3 + \&c.$$

in the particular cases of x which satisfy the equation $\alpha x = x$.

PART II.

I shall now explain another method of deducing the sums of a variety of series, which comprehend amongst them all those which are contained in the former part of this paper; it rests fundamentally on the following formulæ, which have long been known:

$$0 = 1^{2n} - 2^{2n} + 3^{2n} - 4^{2n} + \&c.$$

$$\tfrac{1}{2} = \cos\theta - \cos 2\theta + \cos 3\theta - \&c.$$

$$\tfrac{1}{2} = \cos\theta + \cos 2\theta + \cos 3\theta + \&c.$$

$$0 = 1^{2n+1} - 3^{2n+1} + 5^{2n+1} - \&c.$$

It is unnecessary to give proofs of these and other similar ones which have been frequently noticed, as they may be very easily demonstrated.

Let fx be any function of x developeable in even powers of x, then $\quad f(x) = A + Bx^2 + Cx^4 + Dx^6 + \&c.$

Divide both sides by x^{2k}, then it becomes

$$\frac{f(x)}{x^{2k}} = \frac{A}{x^{2k}} + \frac{B}{x^{2k-2}} + \&c. + K + Lx^2 + Mx^4 + \&c.$$

For x, put successively $1x$, $2x$, $3x$, $4x$, &c. and let the alternate series be taken negatively; these being arranged under each other, we have

$$+\frac{f(x)}{1^{2k}x^{2k}} = +\frac{A}{1^{2k}x^{2k}} + \frac{B}{1^{2k-2}x^{2k-2}} + \&c. + K + Lx^2 1^2 + Mx^4 1^4 +$$

$$-\frac{f(2x)}{2^{2k}x^{2k}} = -\frac{A}{2^{2k}x^{2k}} - \frac{B}{2^{2k-2}x^{2k-2}} - \&c. - K - Lx^2 2^2 - Mx^4 2^4 -$$

$$+\frac{f(3x)}{3^{2k}x^{2k}} = +\frac{A}{3^{2k}x^{2k}} + \frac{B}{3^{2k-2}x^{2k-2}} + \&c. + K + Lx^2 3^2 + Mx^4 3^4 +$$

$$-\frac{f(4x)}{4^{2k}x^{2k}} = -\frac{A}{4^{2k}x^{2k}} - \frac{B}{4^{2k-2}x^{2k-2}} - \&c. - K - Lx^2 4^2 - Mx^4 4^4 -$$

&c. &c. &c.

If we add the vertical columns, we shall have on the left side of the equation the series

$$\frac{1}{x^{2k}}\left\{\frac{f(x)}{1^{2k}} - \frac{f(2x)}{2^{2k}} + \frac{f(3x)}{3^{2k}} - \frac{f(4x)}{4^{2k}} + \&c.\right\}$$

and the right side of the equation consists of three kinds of terms, those which contain negative powers of x, one which does not contain x, and the remaining ones which contain positive powers of x. With respect to these last, they are all of the form $Qx^{2i}\left\{1^{2i} - 2^{2i} + 3^{2i} - 4^{2i} + \&c.\right\}$; and as the series which multiplies Qx^{2i} is equal to zero, all those vertical columns which contain even powers of x will vanish: the term which is independent on x is

$$K - K + K - K + \&c. = \tfrac{1}{2}K$$

and those terms which contain negative powers of x, may be represented by the expression $S\frac{\pm 1}{i^{2k}}$. All the vertical columns being summed, we shall have the equation

$$\frac{f(x)}{1^{2k}} - \frac{f(2x)}{2^{2k}} + \frac{f(3x)}{3^{2k}} - \&c. = AS\frac{\pm 1}{i^{2k}} + BS\frac{\pm 1}{i^{2k-2}} + $$
$$+ CS\frac{\pm 1}{i^{2k-4}} + \&c. + \tfrac{1}{2}K \qquad (A)$$

As the operations by which we have arrived at this expression have been given at length, it will be unnecessary to repeat them with the slight modifications which would be required for cases nearly similar. Thus, if we suppose the function fx developeable according to the odd powers of x, and if we divide both sides of the equation by x^{2k+1} and repeat the same process we have already explained, we shall arrive at the following theorem:

$$\frac{f(x)}{1^{2k+1}} - \frac{f(2x)}{2^{2k+1}} + \frac{f(3x)}{3^{2k+1}} - \frac{f(4x)}{4^{2k+1}} + \&c. =$$

$$= AxS\frac{\pm 1}{i^{2k}} + Bx^3S\frac{\pm 1}{i^{2k-2}} + \&c. + \tfrac{1}{2}K \qquad \text{(B)}$$

Let $f(\theta)$ be any function of θ developeable in the form

$$f(\theta) = A + B\cos\theta + C\cos 2\theta + D\cos 3\theta \ \&c.$$

a very similar process to that which has been already explained will give the

$$f(\theta) - f(2\theta) + f(3\theta) - \&c. = \frac{f(o)}{2} \qquad \text{(C)}$$

and if $f(\theta) = A\cos\theta + B\cos 2\theta + C\cos 3\theta + \&c.$ a similar course will produce the equation

$$f(\theta) + f(2\theta) + f(3\theta) + \&c. = -\tfrac{1}{2}f(o) \qquad \text{(D)}$$

If a function is developeable in even powers of x, then its second function is developeable in the same manner, and so are all its higher functions; therefore if f_1 and f are two functions developeable in even powers of x, such that

$$fx = A^1 + B^1x^2 + C^1x^4 + \&c.$$

$$f_1f^nx = A + Bx^2 + Cx^4 + \&c.$$

Then (A) will become

$$\frac{f_1f^n(x)}{1^{2k}} - \frac{f_1f^n(2x)}{2^{2k}} + \frac{f_1f^n(3x)}{3^{2k}} - \&cc. =$$

$$= AS\frac{\pm 1}{i^{2k}} + BS\frac{\pm 1}{i^{2k-2}} + \&c. + \tfrac{1}{2}K \qquad \text{(E)}$$

These theorems marked (A), (B), (C), and (D), although they possess a very great degree of generality, are not entirely without restriction; it appears at first sight that they are applicable to *all* functions which have the prescribed condition of being expansible in even powers of the variable; such was my opinion of them when I first discovered them; but several results which were evidently incorrect, soon convinced me that some limitation existed, of whose nature I was not aware: it was not until some years after, that I found out the cause of the fallacies which had perplexed me; and still more recently, I discovered that the series on whose sum their truth or falsehood depended, might be expressed by a definite integral. By applying the criterion, which I shall presently explain, we cut off a great variety of series whose sums are erroneously given by the method in question; whether this criterion does not exclude some series whose sums are correctly given, is a point which I do not consider yet completely decided; the difficulties to which the application of acknowledged principles have in this instance conducted us, appear worthy of the attention of mathematicians. A more strict method might have been pursued in determining the sum of that part of the series which is neglected; but this in general leads to such differential equations, as cannot afford us much assistance. I have, however, given one example of this method, and I have shown that when the part which had been neglected, as being apparently equal to zero (but which is in fact a finite quantity,) is added to the sum furnished by the *method of expanding horizontally and summing vertically*, the true value of the series results. This confirms the explanation I have given of the reason of the apparent failure of that method.

It will be sufficient to point out the cause which leads to error, and to determine the conditions on which its existence depends for one only of the series; suitable modifications of the reasoning will readily suggest themselves for the others. I shall therefore, at present, consider the theorem (A). If we turn to the process employed in its investigation, we may remark, that the vertical column $Lx^2(1^2 - 2^2 + 3^2 - 4^2 + \&c.)$ has been neglected, because the series which enters into it as a factor is equal to zero; so also the vertical column $Mx^4(1^4 - 2^4 + 3^4 - \&c.)$ is neglected for the same reason, and similarly for all the remaining vertical columns. Now, although it would be perfectly correct to omit any one, or even any finite number of these vertical columns, as being multiplied by a factor equal to zero, yet it is not legitimate to neglect an infinite number of terms, each multiplied by zero, unless it can be proved that the sum of all the terms so multiplied is not an infinite quantity: this, then, is the latent cause of the false results at which I arrived at the commencement of these enquiries. I shall now explain how they may be obviated, or rather how to assign the condition on which the truth of the theorems just deduced depend. We have considered the series of terms

$$Lx^2(1^2 - 2^2 + 3^2 - \&c.) + Mx^4(1^4 - 2^4 + 3^4 - \&c.) +$$
$$+ Mx^6(1^6 - 2^6 + 3^6 - \&cc.) + \&c.$$

as equal to zero. Any one of the series which here multiply the powers of x, may be considered as arising from the series

$$1^{2n}y - 2^{2n}y^2 + 3^{2n}y^3 + \&c.$$

When $y = 1$, call this series $K_n(y)$, and instead of making $y = 1$, let $y = 1 + o$, which differs from unity by the infinitely small quantity o; then we shall have

$$K_n(1 + o) = c_{1,n} + c_{2,n}o + c_{3,n}o^2 + \&c.$$

Where

$$c_{1,n} = o = 1^{2n} - 2^{2n} + 3^{2n} - \&c. \quad c_{2,n} = 1^{2n+1} - 2^{2n+1} + 3^{2n+1} - \&c.$$

by substituting this value of $K_n(1 + o)$ in the series we had neglected, we shall find

$$\begin{aligned} &\left\{Lx^2c_{1,1}+Mx^4c_{1,2}+Nx^6c_{1,3}+\&c.\right\}\\ +o&\left\{Lx^2c_{2,1}+Mx^4c_{2,2}+Nx^6c_{2,3}+\&c.\right\}\\ +o^2&\left\{Lx^2c_{3,1}+Mx^4c_{3,2}+Nx^6c_{3,3}+\&c.\right\}\\ +o^3&\left\{Lx^2c_{4,1}+Mx^4c_{4,2}+Nx^6c_{4,3}+\&c.\right\}\\ +&\quad\&c. \qquad\qquad \&c. \end{aligned}$$

The first line vanishes on account of the value of $c_{1,n}$, and since o is an infinitesimal, the second line will be larger than the sum of all the rest, provided the multiplicators of the powers of o are finite; if therefore the series $Lx^2c_{2,1}+Mx^4c_{2,2}+Nx^6c_{2,3}+$ &c. is finite since it is multiplied by o, we may neglect the whole of the above expression: our next step must be to determine whether the series

$$Lx^2\left\{1^3-2^3+3^3-\&c.\right\}+Mx^4\left\{1^5-2^5+3^5-\&c.\right\}+$$
$$+Nx^6\left\{1^7-2^7+3^7-\&c.\right\}+\&c.$$

is finite or infinite. It has been observed by EULER, that the following relations exist between the direct and reciprocal powers of the natural numbers.

$$\begin{aligned} 1-2+3-4+\&c. &= +2\tfrac{1}{\pi^2}\left\{1+\tfrac{1}{3^2}+\tfrac{1}{5^2}+\&c.\right\}\\ 1^3-2^3+3^3-4^3+\&c. &= -2\tfrac{1.2.3}{\pi^4}\left\{1+\tfrac{1}{3^4}+\tfrac{1}{5^4}+\&c.\right\}\\ 1^5-2^5+3^5-4^5+\&c. &= +2\tfrac{1...5}{\pi^6}\left\{1+\tfrac{1}{3^6}+\tfrac{1}{5^6}+\&c.\right\}\\ 1^7-2^7+3^7-4^7+\&c. &= -2\tfrac{1...7}{\pi^8}\left\{1+\tfrac{1}{3^8}+\tfrac{1}{5^8}+\&c.\right\}\\ \&c. \qquad\qquad & \qquad\qquad \&c. \end{aligned}$$

These latter being substituted for their equals, we have

$$-2Lx^2\tfrac{1.2.3}{\pi^4}\left(\tfrac{1}{1^4}+\tfrac{1}{3^4}+\tfrac{1}{5^4}+\&c.\right)+2Mx^4\tfrac{1.2.3.4.5}{\pi^6}\left(\tfrac{1}{1^6}+\tfrac{1}{3^6}+\right.$$
$$\left.+\tfrac{1}{5^6}\ \&c.\right)-2Nx^6\tfrac{1..7}{\pi^8}\left(\tfrac{1}{1^8}+\tfrac{1}{3^8}+\tfrac{1}{5^8}+\&c.\right)+\&c.$$

The series which now multiplies each term is in all cases less than 2, consequently this expression will be finite or infinite, according as the series.

$$-Lx^2\frac{1.2.3}{\pi^4}+Mx^4\frac{1...5}{\pi^6}-Nx^6\frac{1....7}{\pi^8}+\&c.$$

is so or the contrary: the product $1.2...n$ may be expressed by means of a definite integral, thus:

$$1.2...n=\int dv\left(\log\frac{1}{v}\right)^n \qquad \begin{bmatrix}v=0\\v=1\end{bmatrix}$$

These products being replaced by the integral, we have

$$\frac{1}{\pi^2}\int dv\left\{-L\left(\frac{x}{\pi}\right)^2\left(\log\frac{1}{v}\right)^3+M\left(\frac{x}{\pi}\right)^4\left(\log\frac{1}{v}\right)^5-N\left(\frac{x}{\pi}\right)^6\left(\log\frac{1}{v}\right)^7+\&c.\right\} \quad \begin{bmatrix}v=0\\v=1\end{bmatrix}$$

$$=\frac{1}{\pi^2}\int dv\log\frac{1}{v}.\left\{-L\left(\frac{x}{\pi}\log\frac{1}{v}\right)^2+M\left(\frac{x}{\pi}\log\frac{1}{v}\right)^4-N\left(\frac{x}{\pi}\log\frac{1}{v}\right)^6+\&c.\right\}$$

Now in order to determine the sum of this series, which evidently depends on the function $f(x)$, let us assume

$$\frac{f(x)}{x^{2k}}-\frac{A}{x^{2k}}-\frac{B}{x^{2k-2}}-\&c.-K=\chi(x)$$

then we have

$$\chi(x)=Lx^2+Mx^4+Nx^6+\&c.$$

And if we put $\frac{x\sqrt{-1}}{\pi}\log\frac{1}{v}$ instead of x, it becomes

$$\chi\left(\frac{x\sqrt{-1}}{\pi}\log\frac{1}{v}\right)=-L\left(\frac{x}{\pi}\log\frac{1}{v}\right)^2+M\left(\frac{x}{\pi}\log\frac{1}{v}\right)^4-N\left(\frac{x}{\pi}\log\frac{1}{v}\right)^6+\&c.$$

And the sum of the series in question is

$$\frac{1}{\pi^2}\int dv\log\frac{1}{v}\cdot\chi\left(\frac{x\sqrt{-1}}{\pi}\log\frac{1}{v}\right) \qquad \begin{bmatrix}v=0\\v=1\end{bmatrix}$$

If therefore this definite integral is finite, the theorem (A) will give correct results. A more convenient form for integration may however be obtained by the following consideration; the series

$$L\frac{x^2}{\pi^2}.1.2.3+M\frac{x^4}{\pi^4}1...5+N\frac{x^6}{\pi^6}1...7+\&c.$$

will always be finite, if the following series

$$A+B\left(\frac{x}{\pi}\right)^2 1.2.3+C\left(\frac{x}{\pi}\right)^4 1.2..5+D\left(\frac{x}{\pi}\right)^6 1.2..7+\&c.$$

is finite, because this series when prolonged to the terms

$Mx^n + Nx^{n+1} +$ &c. will have its terms each greater than the corresponding terms of the other series. Now this series is equal to

$$\int dv \, f\left(\frac{x\sqrt{-1}}{\pi} \log \frac{1}{v}\right) \qquad \begin{bmatrix} v = o \\ v = 1 \end{bmatrix} \qquad \text{(F)}$$

I shall now apply some of the theorems to the investigation of the sums of series, and then explain a method which (when the equations to which it leads can be solved) will in all cases render them correct. And first let $f(\theta) = (\cos\theta)^n = \left(1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \text{\&c.}\right)^n$ this series is capable of being expanded into another, which also proceeds according to the even powers of θ; first let $k = o$, then comparing this with (A), we have

$$\frac{1}{2} = (\cos\theta)^n - (\cos 2\theta)^n + (\cos 3\theta)^n - \text{\&c.} \qquad (3{,}9)$$

Let us now examine if the definite integral is finite, it is in this case.

$$\int dv\left\{\cos\frac{\theta\sqrt{-1}}{\pi}\log\frac{1}{v}\right\}^n = \int dv\left\{\frac{\varepsilon^{-\frac{\theta}{\pi}\log\frac{1}{v}} + \varepsilon^{\frac{\theta}{\pi}\log\frac{1}{v}}}{2}\right\}^n =$$

$$= \int dv\left(\frac{v^{\frac{\theta}{\pi}} + v^{-\frac{\theta}{\pi}}}{2}\right)^n$$

$$= \frac{1}{2^n}\left\{\frac{1}{1 - n\frac{\theta}{\pi}} + \frac{n}{1}\cdot\frac{1}{1 + (2-n)\frac{\theta}{\pi}} + \frac{n.n-1}{1.2}\cdot\frac{1}{1 + (4-n)\frac{\theta}{\pi}} + \text{\&c.}\right\} \qquad \begin{bmatrix} v = o \\ v = 1 \end{bmatrix}$$

If n is a whole positive number we already know that the series (3,9) is correct; if n is a fraction the series which expresses the value of the definite integral is finite for all values of θ, except such as are contained in $\theta = -\frac{\pi}{2i - n}$ i being any whole number; if n is negative, then the series is finite for all positive values of θ; it appears then that whatever be the value of n if θ is positive, we have

$$\frac{1}{2} = \frac{1}{(\cos\theta)^n} - \frac{1}{(\cos 2\theta)^n} + \frac{1}{(\cos 3\theta)^n} - \&c.$$

From the theorem (A) we may readily determine the value of the series

$$\frac{1}{1^{2k}(\cos\theta)^n} - \frac{1}{2^{2k}(\cos 2\theta)^n} + \frac{1}{3^{2k}(\cos 3\theta)^n} - \&c.$$

For let $f(\theta) = \frac{1}{(\cos\theta)^n} = A' + A'_n\theta^2 + B'_n\theta^4 + C'_n\theta^6 + \&c.$

And the sum of the series required will be

$$S\frac{\pm 1}{i^{2k}} + A'_n\theta^2 S\frac{\pm 1}{i^{2k-2}} + B'_n\theta^4 S\frac{\pm 1}{i^{2k-4}} + \&c. + \theta^{2k}K'_n.\frac{1}{2}$$

Which is precisely the same sum as we have already found in (2,4), except that we now find that it applies to fractional or surd values of n, as well as to whole numbers.

Let us next suppose $f(\theta) = (\tan\theta)^{2l+1} = T_1\theta^{2l+1} + T_3\theta^{2l+3} + T_5\theta^{2l+5} + \&c.$ this give the series

$$\frac{(\tan\theta)^{2l+1}}{1^{2k+1}} - \frac{(\tan 2\theta)^{2l+1}}{2^{2k+1}} + \frac{(\tan 3\theta)^{2l+1}}{3^{2k+1}} - \&c. =$$

$$= T_1\theta S\frac{\pm 1}{i^{2(k-l)}} + T_3\theta^3 S\frac{\pm 1}{i^{2(k-l-1)}} + \&c.$$

The definite integral in this case being

$$\int dv \left\{\tan\frac{\theta\sqrt{-1}}{\pi}\log\frac{1}{v}\right\}^{2l+1} = \frac{1}{(\sqrt{-1})^{2l+1}}\int dv\left\{\frac{-1+v^{2\frac{\theta}{\pi}}}{1+v^{2\frac{\theta}{\pi}}}\right\}^{2l+1} \quad \begin{bmatrix} v=0 \\ v=1 \end{bmatrix}$$

$$= \frac{1}{(\sqrt{-1})^{2l+1}}\left\{A + \frac{B}{1+2\frac{\theta}{\pi}} + \frac{C}{1+4\frac{\theta}{\pi}} + \frac{D}{1+6\frac{\theta}{\pi}} + \&c.\right\}$$

Since $\frac{-1+v^{2\frac{\theta}{\pi}}}{1+v^{2\frac{\theta}{\pi}}} = A + Bv^{2\frac{\theta}{\pi}} + Cv^{4\frac{\theta}{\pi}} + \&c.$ this is always finite if θ is positive, because it is less than $A+B+C+\&c.$ which is equal to zero.

In the former part we could only determine the value of the series

$$\frac{1}{(\sin\theta)^n}-\frac{1}{(\sin 2\theta)^n}+\frac{1}{(\sin 3\theta)^n}-\&c.$$

when n is an even number, nor will the method now employed enable us to find its value when n is odd; the reason of this is that

$$(\sin\theta)^{-n}=\left(\frac{\theta}{1}-\frac{\theta^3}{1.2.3}+\frac{\theta^5}{1.2.3.4.5}-\&c.\right)^{-n}$$

is developable in a series proceeding according to the even powers of θ only when n is an even number, if n is odd, it proceeds according to the odd powers.

The theorems contained in this second part are applicable to a very extensive class of series which have not, I believe, yet been considered. In (A) let $f(\theta)=\cos^2\theta=\cos(\cos\theta)$ and putting $k=o$ we have

$$\frac{\cos 1}{2}=\cos^2\theta-\cos^2 2\theta+\cos^2 3\theta-\&c. \qquad (4,1)$$

The definite integral which is the criterion of the truth of this value, is

$$\int dv\cos\frac{v^{\frac{\theta}{\pi}}+v^{-\frac{\theta}{\pi}}}{2}=A+\frac{B}{1+2\frac{\theta}{\pi}}+\frac{C}{1+4\frac{\theta}{\pi}}+\frac{D}{1+6\frac{\theta}{\pi}}+\&c.\quad\begin{bmatrix}v=o\\v=1\end{bmatrix}$$

$$+\frac{B}{1-2\frac{\theta}{\pi}}+\frac{C}{1-4\frac{\theta}{\pi}}+\frac{D}{1-6\frac{\theta}{\pi}}+\&c.$$

And this is always finite unless θ is an even submultiple of π; if we make $k=1$ and $k=2$, we shall have the following theorems, which are true with the same restrictions.

$$\cos 1.\,S\frac{\pm 1}{i^2}+\frac{\sin 1}{2}\cdot\frac{\theta^2}{2}=\frac{\cos^2\theta}{1^2}-\frac{\cos^2 2\theta}{2^2}+\frac{\cos^2 3\theta}{3^2}-\&c. \qquad (4,2)$$

$$\cos 1.\,S\frac{\pm 1}{i^4}+\theta^2\frac{\sin 1}{2}S\frac{\pm 1}{i^2}-\frac{\theta^4}{2}\left(\frac{\sin 1}{1.2.3.4}+\frac{\cos 1}{8}\right)=$$

$$=\frac{\cos^2\theta}{1^4}-\frac{\cos^2 2\theta}{2^4}+\frac{\cos^2 3\theta}{3^4}-\&c. \qquad (4,3)$$

If $f(\theta)=(\cos{}^n\theta)^m$, since this is capable of being developed in a series proceeding according to the even powers of θ, we shall have if $k = o$

$$\frac{(\cos^{n-1} 1)^m}{2} = (\cos {}^n\theta)^m - (\cos {}^n 2\theta)^m + (\cos {}^n 3\theta)^m - \&c. \qquad (4,4)$$

and the definite integral is finite in this case, whenever θ and π are incommensurable: we may therefore in the same circumstance have the value of the series

$$\frac{(\cos {}^n\theta)^m}{1^{2k}} - \frac{(\cos {}^n 2\theta)^m}{2^{2k}} + \frac{(\cos {}^n 3\theta)^m}{3^{2k}} - \&c.$$

The theorems marked (A) and (B) in this paper correspond with that marked (12) in Mr. HERSCHEL's memoir "On various points of Analysis," printed in the Philosophical Transactions for 1814; with the first of these it coincides when n is an even number, and with the second when it is an odd one: the theorem alluded to is

$$S\left\{\frac{(-1)^{x+1}}{x^n} \cdot \frac{f(\varepsilon^{x\theta}) + (-1)^n f(\varepsilon^{-x\theta})}{2}\right\} = {}^0L(2) . \alpha_n \theta^n + {}^2L(2) . \alpha_{n-2} \theta^{n-2} + \&c.$$

where ${}^0L = 1 - 1 + 1 - \&c. = \frac{1}{2}\ {}^{2x}L(2) = \frac{(2^{2x-1}-1)\pi^{2x}}{1.2\ldots 2x} B_{2n-1}$

Now this latter expression is the value of those series which I have expressed by $S\frac{\pm 1}{i^{2k}}$. Both methods give the same result, and as that result is very frequently erroneous, I shall confirm the truth of the explanation I have offered, by shewing in a particular case, that if the sum of that part of the series which had been neglected as being equal to zero, is found and added to the other part, the result will no longer be erroneous: the example I shall examine is the series

$$\frac{1}{1+\theta^2} - \frac{1}{1+2^2\theta^2} + \frac{1}{1+3^2\theta^2} - \frac{1}{1+4^2\theta^2} + \&c.$$

In Mr. HERSCHEL's theorem, making $f(\varepsilon^t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \&c. = \frac{1+\theta t\sqrt{-1}}{1+\theta^2 t^2}$ we have $\alpha_0 = 1$, and the equation becomes

$$S\left\{\frac{(-1)^{x+1}\left\{\frac{1+x\theta\sqrt{-1}}{1+x^2\theta^2}+\frac{1-x\theta\sqrt{-1}}{1+x^2\theta^2}\right\}}{2}\right\}={}^{\circ}L(2).1$$

or $\quad S(-1)^{x+1}\frac{1}{1+x\theta^2}=\frac{1}{2}$ or

$$\frac{1}{2}=\frac{1}{1+\theta^2}-\frac{1}{1+2^2\theta^2}+\frac{1}{1+3^2\theta^2}-\&c. \qquad (4,4)$$

The same series being summed by the theorem (A) in this paper gives the same result; but then the theorem alluded to only declares this to be the true value in case a certain series is finite, which series is

$$+\theta^2.1.2.3+\theta^4.1.2.3.4.5+\theta^6.1.2..7+\&c.$$

but this series can only be finite when θ is actually equal to zero: the method which I have explained in this paper points out that the equation

$$\frac{1}{2}=\frac{1}{1+\theta^2}-\frac{1}{1+2^2\theta^2}+\frac{1}{1+3^2\theta^2}-\&c.$$

can only be depended on when $\theta=o$, in that case it is known to be correct. I have already stated that the reason why the value of the series so found is incorrect, is that the series

$$-\theta^2(1^2-2^2+\&c.)+\theta^4(1^4-2^4+\&c.)-\theta^6(1^6-2^6+\&c.)+\&c.$$

has been neglected because the coefficient of each term is zero. I shall now proceed to investigate the sum of this series, and shall prove that it is equal to a finite function of θ: let

$$y=c^2(1^2\varepsilon^x-2^2\varepsilon^{2x}+\&c.)+c^4(1^4\varepsilon^x-2^4\varepsilon^{2x}+\&c.)+\&c.$$

then y is equal to the sum of the series whose value we are seeking; if $c=\theta\sqrt{-1}$ and $x=o$, differentiate y twice relative to x, and multiply by c^2, and we find

$$c^2\frac{d^2y}{dx^2}=c^4(1^4\varepsilon^x-2^4\varepsilon^{2x}+\&c.)+c^6(1^6\varepsilon^x-2^6\varepsilon^{2x}+\&c.)+\&c.$$

Hence the equation for determining y is

$$c^2\frac{d^2y}{dx^2}-y=-c^2(1^2\varepsilon^x-2^2\varepsilon^{2x}+\&c.)=-c^2\frac{\varepsilon^x-\varepsilon^{2x}}{(1+\varepsilon^x)^3}$$

And the value of y is

$$y = \frac{c}{2}\left\{ \varepsilon^{\frac{x}{c}} \int \varepsilon^{-\frac{x}{c}} \frac{\varepsilon^{2x} - \varepsilon^{x}}{(1+\varepsilon^{x})^{3}}\, dx - \varepsilon^{-\frac{x}{c}} \int \varepsilon^{\frac{x}{c}} \frac{\varepsilon^{2x} - \varepsilon^{x}}{(1+\varepsilon^{x})^{3}} \right\}$$

These integrals must be taken between the limits $x = -\infty$ and $x = o$, putting $\varepsilon^{x} = v$ this equation is changed into

$$y = \frac{c}{2}\left\{ v^{\frac{1}{c}} \int \frac{v-1}{(1+v)^{3}}\, v^{-\frac{1}{c}}\, dv - v^{-\frac{1}{c}} \int \frac{v-1}{(1+v)^{3}}\, v^{\frac{1}{c}}\, dv \right\}$$

where the limits of v are $v = o$ $v = 1$, in the latter integral put $v = \frac{1}{u}$, and we have

$$y = \frac{c}{2}\left\{ v^{\frac{1}{c}} \int \frac{v-1}{(1+v)^{3}}\, v^{-\frac{1}{c}}\, dv + u^{\frac{1}{c}} \int \frac{u-1}{(1+u)^{3}}\, u^{-\frac{1}{c}}\, du \right\} \quad \left\{ \begin{matrix} v = o & u = 1 \\ v = 1 & u = \infty \end{matrix} \right\}$$

but this is equal to $\frac{c}{2} v^{\frac{1}{c}} \int \frac{v-1}{(1+v)^{3}}\, v^{-\frac{1}{c}}\, dv$ between the limits $v = o$ and $v = \infty$, which is equal to $-\frac{\pi}{2c \sin \frac{\pi}{c}}$ hence

$$y = -\frac{\pi}{2c \sin \frac{\pi}{c}} = c^{2}(1^{2} - 2^{2} + \&c.) + c^{4}(1^{4} - 2^{4} + \&c.) + + c^{6}(1^{6} - 2^{6} + \&c.) + \&c.$$

If $c = \theta\sqrt{-1}$ we have

$$-\frac{\pi}{\theta\left\{ \varepsilon^{\frac{\pi}{\theta}} - \varepsilon^{-\frac{\pi}{\theta}} \right\}} = -\theta^{2}(1^{2} - 2^{2} + \&c.) + \theta^{4}(1^{4} - 2^{4} + \&c.) - -\theta^{6}(1^{6} - 2^{6} + \&c.) + \&c.$$

This being added to $\frac{1}{2}$ the value given by the theorem (A) produces

$$\frac{1}{2} - \frac{\pi}{\theta\left\{ \varepsilon^{\frac{\pi}{\theta}} - \varepsilon^{-\frac{\pi}{\theta}} \right\}} = \frac{1}{1+\theta^{2}} - \frac{1}{1+2^{2}\theta^{2}} + \frac{1}{1+3^{2}\theta^{2}} - \frac{1}{1+4^{2}\theta^{4}} + \&c.$$

which is the same value that EULER had assigned to this series.

From the value which has been found for y, or for the series

$$c^{2}(1^{2} - 2^{2} + \&c.) + c^{4}(1^{4} - 2^{4} + \&c.) + \&c.$$

I am inclined to conclude that although the series $1^{2n} - 2^{2n} +$

$3^{2n} -$ &c. is equal to zero for any finite value of n, yet that when n is infinite, the sum of this series is also infinite.

It was my intention to have produced from several of the series whose sums have been found in this Paper, the values of several continued products; but the length to which it has already extended will prevent me from more than merely noticing, that many very curious ones will present themselves by integrating the series whose sums have been given.

Since this Paper was written, in a conversation with M. POISSON, I mentioned one of the series which it contained, and remarked, that the principle employed led to many erroneous results; that gentleman observed, that many years before he had been led to series nearly similar, in endeavouring to integrate the equations representing the planetary motions, by means of series arranged according to some other functions of the time than the usual ones of the sines and cosines: he obligingly showed me some of his papers relating to this subject, in one of which was a series which in a particular case became the one I had mentioned; the mode of investigation by which he had arrived at these series he had however laid aside, because it rested on the sums of the diverging series $1^{2n} - 2^{2n} + 3^{2n} -$ &c. on which he observed we cannot depend. To the same distinguished analyst I am indebted for some farther information on this subject. M. POISSON was one of the commissioners appointed by the Institute of France to examine the manuscripts which were left by LAGRANGE, amongst these was one entitled "a method of summing series," which depended on the values of the same diverging series as those used by M. POISSON and myself; unfortunately it is very short, and its illustrious author does not appear to have resumed the subject: possibly the erroneous values which it gives for the sum of certain series might have caused him to reject it.

C. BABBAGE.

March 25, 1819.